Multiscale Modelling Applied to Fractured Oil Shale Rock

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Motivation

Example of highly heterogeneous medium we use into the diffusion equation:

$$\nabla \cdot (K(x) \nabla u(x)) = f(x)$$

Permeability Field from SPE-10.
Motivation

Example of highly heterogeneous medium we use into the diffusion equation:

\[ \nabla \cdot (K(x) \nabla u(x)) = f(x) \]
Motivation

Heterogeneity built on the Computational Grid
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Heterogeneity built on the Computational Grid
Motivation

Computational Grid with Simplified Geometries
Outline

• Analytical Approach
• Zero\textsuperscript{th}-order Approximation for Diffusion Equations - Upscaled Coefficient
• Comparison with Theoretical and Numerical Results
• First-order Approximation for Diffusion Equations.
• Results of Convergence Linear and Nonlinear
• Analytical Upscaling for Generalized Geometries
• Application to Transport on Highly Heterogeneous Media
• Conclusion and Future Work
Each isotropic and heterogeneous media can be formalized

\[
K_s^\varepsilon (x) = \begin{cases} 
\xi_1 & x \in \Omega_c^\varepsilon \\
\xi_2 & x \in \Omega^\varepsilon \setminus \Omega_c^\varepsilon
\end{cases}
\]

with $\xi_1$ at the inclusion $\Omega_c^\varepsilon$, and $\xi_1 : \xi_2$ is the ratio: 10:1, 100:1, 1000:1 (for example).

Where $\varepsilon = l/L$, $l$ is the length of the inclusion and $L$ the domain size.
Analytical Approach

Each isotropic and heterogeneous media can be formalized

\[ K^\varepsilon_s(x) = \begin{cases} \xi_1 & x \in \Omega^\varepsilon_c \\ \xi_2 & x \in \Omega^\varepsilon \setminus \Omega^\varepsilon_c \end{cases} \]

with \( \xi_1 \) at the inclusion \( \Omega^\varepsilon_c \), and \( \xi_1 : \xi_2 \) is the ratio: 10:1, 100:1, 1000:1 (for example).

Where \( \varepsilon = l/L \), \( l \) is the length of the inclusion and \( L \) the domain size.

**Goal:** To propose an Upscaled Darcy’s Law \( q^0(x) \), the limit as \( \varepsilon \to 0 \):

\[ q^\varepsilon(x) = -\frac{K^\varepsilon_s(x) k_r(u^\varepsilon(x))}{\rho g} \nabla u^\varepsilon(x) = -K^\varepsilon(x, u^\varepsilon(x)) \nabla u(x) \simeq -K^0(u^0) \nabla u^0(x) = q^0(x) \]

\[ \nabla \cdot q^\varepsilon(x) = f(x) \simeq \nabla \cdot q^0(x) = f(x) \]
Sequence of $K^\varepsilon(x)$, as $\varepsilon \to 0$

\[
\nabla \cdot (K^\varepsilon(x) \nabla u^\varepsilon(x)) = f(x)
\]

Highly oscillating media $K^\varepsilon(x)$, as $\varepsilon \to 0$
Sequence of $K^\varepsilon(x)$, as $\varepsilon \to 0$

$$\nabla \cdot (K^\varepsilon(x) \nabla u^\varepsilon(x)) = f(x)$$

Highly oscillating media as $\varepsilon \to 0$
Diffusion in Periodic Media


Consider the family of BVP’s

\[
\begin{aligned}
\nabla \cdot (K^\varepsilon (x) \nabla u^\varepsilon (x)) &= f(x) \quad x \in \Omega \\
\n\nabla u^\varepsilon (x) &= g(x) \quad x \in \partial \Omega
\end{aligned}
\]

Let \( y = \varepsilon^{-1} x \), and the expansion:

\[ u^\varepsilon (x) \approx u^0 (x, y) + \varepsilon u^1 (x, y) \]

**Definition 1** \( H \)-Convergence A constant matrix \( K^0 \) is said to be the homogenized limit of \( K^\varepsilon \), if and only if, in the limit as \( \varepsilon \to 0 \):

\[
\begin{aligned}
u^\varepsilon &\to u^0 \quad \text{in} \quad L^2 (\Omega) \\
K^\varepsilon (x) \nabla u^\varepsilon (x) &\rightharpoonup K^0 \nabla u^0 (x) \quad \text{in} \quad L^2 (\Omega) \\
K^0_{ij} &= \int_Y K(y) (\delta_{i,j} + \partial_{y_i} w_i(y)) \, dy
\end{aligned}
\]

And \( w_i \in H^1(Y) \) is the periodic solution of

\[ \nabla \cdot (K(y) \nabla w_i(y)) = \nabla \cdot (K(y)e_i) \]
Known results for the n-dimensional case

If $K(x)$ is a Layered Media and/or Slanted Layered Media

Then $K^0$ is the arithmetic and harmonic averages layered and its rotation matrix.
Some Review on Numerical Methods

- Durlofsky et al. Numerical Calculation of Equivalent Grid Block Permeability Tensor (1991 and on)
  **Idea** To compute the effective numerically.

- Amaziane et al. Equivalent permeability and simulation of two-phase flow in heterogeneous media
  **Idea** Apply the idea of solving the cell-problem for generalized permeability field, under various BC and show agreement with fine-scale solution (1991) and (2001).

- MsFEM: initially proposed by Hou and Wu (JCP - 1997).
  **Idea**: To construct a finite element basis function, based on a solution of local problem. It has been improved and widely applied. (see Efendiev, Y. for references).

- Black Box Multigrid Numerical Homogenization Algorithm - Moulton, Dendy and Hyman (JCP-1998)
  **Idea**: Multiple length scales are captured by a multilevel iterative solver. It has not been extended to Nonlinear Equations.
Diffusion in Periodic Media

Consider the family of BVP’s
\[
\begin{align*}
\nabla \cdot (K^\varepsilon(x) \nabla u^\varepsilon(x)) &= f(x) \quad x \in \Omega \\
u(x) &= g(x) \quad x \in \partial \Omega
\end{align*}
\]

Let \( y = \varepsilon^{-1}x \), and the expansion:
\[
u^\varepsilon(x) \approx u^0(x, y) + \varepsilon u^1(x, y)
\]

**Definition 2 H-Convergence** A constant matrix \( K^0 \) is said to be the homogenized limit of \( K^\varepsilon \), if and only if, in the limit as \( \varepsilon \to 0 \):
\[
u^\varepsilon \to u^0 \quad \text{in} \quad L^2(\Omega)
\]
\[
K^\varepsilon(x) \nabla u^\varepsilon(x) \rightharpoonup K^0 \nabla u^0(x) \quad \text{in} \quad L^2(\Omega)
\]
\[
K^0_{ij} = \int_Y K(y) (\delta_{i,j} + \partial_{y_i} w_i(y)) \, dy
\]
And \( w_i \in H^1(Y) \) is the periodic solution of
\[
\nabla \cdot (K(y) \nabla w_i(y)) = \nabla \cdot (K(y) e_i)
\]
Result*: $\tilde{w}^1(y) \in L^2(Y)$ approximates $w^1(y) \in H^1(Y)$

The analytical $\tilde{w}^1(y)$, are illustrated below:

And their respective true numerical solution:

4, 8 and 16 square inclusions, respectively.

Analytical Approx. for the Homogenized Coefficient

By assuming that, for each $i = 1, \ldots, n$

$$
\tilde{w}^i(y) = \left( \int_0^{y_i} \frac{d\tau}{K(y_1, y_2, \ldots, \tau, \ldots, y_n)} - y_i \right) / \left( \int_0^1 \frac{d\tau}{K(y_1, y_2, \ldots, \tau, \ldots, y_n)} \right)
$$

One has that:

$$
\tilde{K}^0 = \int_Y diag(R_1, R_2, \ldots R_n) dY
$$

where:

$$
R_i(y) = \frac{1}{\int_0^1 \frac{d\tau}{K(y_1, y_2, \ldots, \tau, \ldots, y_n)}}
$$

One has that:

$$
\tilde{K}^0 = \int_Y diag(R_1, R_2, \ldots R_n) dY
$$

...$\tilde{K}^0$ is the arithmetic average of the harmonic average.
Analytical Approx. for the Homogenized Coefficient

By assuming that, for each \( i = 1, \ldots, n \)

\[
\tilde{w}^i (y) = \left( \int_0^{y_i} \frac{d\tau}{K(y_1, y_2, \ldots, \tau, \ldots, y_n)} - y_i \right)
\]

One has that:

\[
\tilde{K}^0 = \int_Y \text{diag}(R_1, R_2, \ldots R_n) dY
\]

where:

\[
R_i(y) = \frac{1}{\int_0^1 \frac{d\tau}{K(y_1, y_2, \ldots, \tau, \ldots, y_n)}}
\]

One has that:

\[
\tilde{K}^0 = \int_Y \text{diag}(R_1, R_2, \ldots R_n) dY
\]

Result: \( \tilde{K}^0 \) is the lower bound of the Generalized Voig-Reiss Inequality:

\[
\tilde{K}^0 \leq K^0 \leq K^u
\]

where \( K^u = \left( \int_Y \frac{dy_j}{\int_Y K(y)dy_i} \right)^{-1} \), with \( j \neq i \) (Jikov et al. (1994)).
Defining a Corrector to $\tilde{K}^0$

Let $\frac{\partial \tilde{w}^j}{\partial y_i} = 0$ a.e. in $Y$, for $i \neq j$, and define $C_{ij}^\varepsilon(x) = C_{ij}(y)$,

$$C_{ij}^\varepsilon(x) = \delta_{ij} + \frac{\partial \tilde{w}^j}{\partial y_i} = \frac{1}{K(y)} \left( \int_0^1 \frac{d\tau}{K(y_1, \ldots, \tau, \ldots, y_n)} \right)^{-1}$$

$$\nabla u^\varepsilon(x) = C^\varepsilon(x) \nabla u^0(x) + h.o.t$$

We define $K^0$, a corrector of $\tilde{K}^0$, as:

$$K^0 = ||C_{ii}^\varepsilon||_2 \tilde{K}^0 = C_{ii} \tilde{K}^0$$

And the approximate solution has being corrected by a factor $C$.

* Sviercoski and Travis in Transport in Porous Media, (submitted)
First-Order and Upper Bound Estimate for the Error

The first order approximation comes readily as:

\[ u^\varepsilon(x) \simeq u^0(x) + \sum_i C_{ii} \tilde{w}_i \frac{\partial u^0}{\partial x_i} \]

And the Error for the zeroth order approximation:

\[ E = \| u^\varepsilon(x) - u^0(x) \|_2 \leq \left\| \sum_i C_{ii} \tilde{w}_i \frac{\partial u^0}{\partial x_i} \right\|_2 = UBE \]

And,

\[ \nabla u^\varepsilon(x) \simeq [\delta_{i,j} + \sum_{i=1}^n C_{ii} \nabla y \tilde{w}_i(y)] \frac{\partial u^0}{\partial x_i} + \ldots \]

\[ \nabla u^\varepsilon(x) \simeq P^\varepsilon(x) \nabla u^0(x) + \varepsilon \sum_{i=1}^n \tilde{w}_i(y) \nabla \left( \frac{\partial u^0}{\partial x_i} \right) + \ldots \]

Comparing with Theoretical Results

For the checkerboard case, the geometric average is obtained:

<table>
<thead>
<tr>
<th>$\xi_1 : \xi_2$</th>
<th>$K^h$</th>
<th>$K^g$</th>
<th>$C\tilde{K}^0 = K^0$</th>
<th>$K^u$</th>
<th>$K^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5:20</td>
<td>8.02</td>
<td>10</td>
<td>$1.0725 \times 9.31 = 9.98$</td>
<td>10.79</td>
<td>12.56</td>
</tr>
<tr>
<td>1:10</td>
<td>1.82</td>
<td>3.16</td>
<td>$1.1732 \times 2.60 = 3.05$</td>
<td>3.94</td>
<td>5.53</td>
</tr>
<tr>
<td>2:8</td>
<td>3.21</td>
<td>4</td>
<td>$1.0725 \times 3.722 = 3.99$</td>
<td>4.35</td>
<td>5.02</td>
</tr>
<tr>
<td>4:16</td>
<td>6.40</td>
<td>8</td>
<td>$1.0725 \times 7.4451 = 7.985$</td>
<td>8.69</td>
<td>10.05</td>
</tr>
<tr>
<td>16:4</td>
<td>6.39</td>
<td>8</td>
<td>$1.0725 \times 7.38 = 7.91$</td>
<td>8.64</td>
<td>10.00</td>
</tr>
</tbody>
</table>
## Comparison with Published Numerical Results

![Shapes: Square, Circle, Lozenge](image)

**Figure 1**: Ratio - 10:1 and all have area equals $\frac{1}{4}$

<table>
<thead>
<tr>
<th>Shape</th>
<th>$K^h$</th>
<th>$K^{num}$</th>
<th>$C \tilde{K}^0 = K^0$</th>
<th>$K^u$</th>
<th>$K^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>square</td>
<td>1.3</td>
<td>1.548</td>
<td>$1.0937 \times 1.4091 = 1.5411$</td>
<td>1.695</td>
<td>3.2952</td>
</tr>
<tr>
<td>circle</td>
<td>1.29</td>
<td>1.516</td>
<td>$1.08 \times 1.403 = 1.5156$</td>
<td>1.791</td>
<td>3.2511</td>
</tr>
<tr>
<td>lozenge</td>
<td>1.29</td>
<td>1.573</td>
<td>$1.069 \times 1.417 = 1.5148$</td>
<td>1.936</td>
<td>3.2361</td>
</tr>
</tbody>
</table>

Comparison with published numerical results

Figure 2: Tests 1 - 1:10, Test 2 - 1:10 and Test 3 - 1:100 (left) and Test 4 - 1:10 (right)

<table>
<thead>
<tr>
<th>Test</th>
<th>$K^h$</th>
<th>$K#$</th>
<th>$C\tilde{K}^0 = K^0$</th>
<th>$K^u$</th>
<th>$K^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1</td>
<td>3.09</td>
<td>6.52</td>
<td>$1.093 \times 5.91 = 6.459$</td>
<td>7.09</td>
<td>7.75</td>
</tr>
<tr>
<td>Test 2</td>
<td>3.09</td>
<td>6.52</td>
<td>$1.093 \times 5.91 = 6.459$</td>
<td>7.09</td>
<td>7.75</td>
</tr>
<tr>
<td>Test 3</td>
<td>3.89</td>
<td>59.2</td>
<td>$1.1378 \times 51 = 58.03$</td>
<td>67</td>
<td>76.0</td>
</tr>
<tr>
<td>Test 4</td>
<td>1.48</td>
<td>3.106</td>
<td>$1.0663 \times 2.98 = 3.177$</td>
<td>3.27</td>
<td>4.24</td>
</tr>
</tbody>
</table>

Error Analysis

Consider the following BVP in \( \Omega = [0, 1]^2 \):

\[
\begin{aligned}
\nabla \cdot (K^\epsilon(x) \nabla u^\epsilon(x)) &= 1 \quad x \in \Omega \\
u^\epsilon(x) &= 0 \quad x \in \partial \Omega
\end{aligned}
\approx
\begin{aligned}
\nabla \cdot (K^0 \nabla u^0(x)) &= 1 \quad x \in \Omega \\
u^0(x) &= 0 \quad x \in \partial \Omega
\end{aligned}
\]

\( K^\epsilon(x) \) (left), \( u^\epsilon(x) \simeq u^0(x) \) (right)
Consider the following BVP in $\Omega = [0, 1]^2$:

\[
\begin{aligned}
\nabla \cdot (K^\varepsilon(x) \nabla u^\varepsilon(x)) &= 1 & x & \in \Omega \\
u^\varepsilon(x) &= 0 & x & \in \partial \Omega \\
\end{aligned}
\]

\[
\approx \begin{aligned}
\nabla \cdot (K^0 \nabla u^0(x)) &= 1 & x & \in \Omega \\
u^0(x) &= 0 & x & \in \partial \Omega \\
\end{aligned}
\]

\[u^\varepsilon(x) \simeq u^0(x) \text{ (left), } u^\varepsilon(x) \simeq u^0(x) + \sum_i C_{ii} \tilde{w}^i \frac{\partial u^0}{\partial x_i} \text{ (right).}\]
$u^\varepsilon(x)$ fine-scale (solid) and approximations (dashed)

$u^\varepsilon(x) \simeq u^0(x)$ (left), $u^\varepsilon(x) \simeq u^0(x) + \sum_i C_{ii} \tilde{w}^i \frac{\partial u^0}{\partial x_i}$ (right).
$u^\varepsilon(x)$ fine-scale (solid) and approximations (dashed)

$u^\varepsilon(x) \simeq u^0(x)$ (left), $u^\varepsilon(x) \simeq u^0(x) + \sum_i C_{ii} \tilde{w}^i \frac{\partial u^0}{\partial x_i}$ (right).
$u^\varepsilon(x)$ fine-scale (solid) and approximations (dashed)

\[ u^\varepsilon(x) \simeq u^0(x) \text{ (left), } u^\varepsilon(x) \simeq u^0(x) + \sum_i C_{ii} \tilde{w}^i \frac{\partial u^0}{\partial x_i} \text{ (right)}. \]
Ratio 10:1 on $[0, 1]^2$ with $K^0 = 1.0937 \times 1.4091 = 1.5411$

$$E = \left\| u^\varepsilon(x) - u^0(x) \right\|_2 \leq C \left\| \sum_i \tilde{w}_i \frac{\partial u^0}{\partial x_i} \right\|_2 = UBE$$

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$\left| u^\varepsilon - u^0 \right|_2$</th>
<th>UBE</th>
<th>$\left| u^\varepsilon - (u^0 + u^1) \right|_2$</th>
<th>$\left| \nabla u^\varepsilon - P^\varepsilon \nabla u^0 \right|_2$</th>
<th>$\left| K^\varepsilon \nabla u^\varepsilon - K^0 P^\varepsilon \nabla u^0 \right|_2$</th>
<th>#nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0.5)^1$</td>
<td>1.10e-2</td>
<td>1.61e-2</td>
<td>4.43e-3</td>
<td>6.91e-2</td>
<td>2.36e-1</td>
<td>16641</td>
</tr>
<tr>
<td>$(0.5)^2$</td>
<td>4.92e-3</td>
<td>7.48e-3</td>
<td>2.31e-3</td>
<td>5.75e-2</td>
<td>2.11e-1</td>
<td>16641</td>
</tr>
<tr>
<td>$(0.5)^3$</td>
<td>2.13e-3</td>
<td>3.74e-3</td>
<td>1.05e-3</td>
<td>4.53e-2</td>
<td>1.97e-1</td>
<td>16641</td>
</tr>
<tr>
<td>$(0.5)^4$</td>
<td>9.48e-4</td>
<td>1.80e-3</td>
<td>4.80e-4</td>
<td>3.89e-2</td>
<td>1.88e-1</td>
<td>16265</td>
</tr>
<tr>
<td>$(0.5)^5$</td>
<td>5.13e-4</td>
<td>9.33e-4</td>
<td>2.54e-4</td>
<td>3.65e-2</td>
<td>1.87e-1</td>
<td>161393</td>
</tr>
</tbody>
</table>

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Comparison between Gradients of the approximations

\( \nabla u^\varepsilon(x) \) (left) and \( \nabla u^0(x) \) (center) and \( P^\varepsilon(x) \nabla u^0(x) \) (right).
The conductivity is given by the Van Genuchten’s relationship:

\[
K^\varepsilon (x, u_\varepsilon(x)) = K_s^\varepsilon (x) \left( (1 + |\alpha u_\varepsilon(x)|^n)^{-m} \right)^p \left[ 1 - (1 - \left( (1 + |\alpha u_\varepsilon(x)|^n)^{-m} \right)^{\frac{1}{m}} \right]^{2}
\]

and parameters: \( \alpha = 1.04 \, m^{-1} \), \( m = 0.283 \), \( n = \frac{1}{1 - m} \) and \( p = 0.5 \).

* Sviercoski, Popov, Travis in preparation to JCP.
## Numerical Convergence for Nonlinear Equations

Table 1: **Zero-th-order Approximation for $K^\varepsilon(x)$ with circular inclusions, ratio 1:1000; $K^0_s = 1.1866 \times 436.60 = 518.09$.**

| $\varepsilon$ | $||u_\varepsilon - u_0||_2$ | UBE | $||\nabla u_\varepsilon - \nabla u_0||_2$ | $||K^\varepsilon \nabla u_\varepsilon - K^0 \nabla u_0||_2$ | grid |
|----------------|----------------------------|------|--------------------------------|--------------------------------|------|
| (0.5)$^1$      | 2.22e-2                    | 3.65e-2 | 1.58e-1          | 3.10e+0        | 88X88 |
| (0.5)$^2$      | 1.08e-2                    | 2.17e-2 | 1.50e-1          | 2.85e+0        | 94X94 |
| (0.5)$^3$      | 4.00e-3                    | 1.21e-2 | 1.45e-1          | 2.60e+0        | 104X104 |
| (0.5)$^4$      | 3.25e-3                    | 5.92e-3 | 1.41e-1          | 2.47e+0        | 114X114 |
| (0.5)$^5$      | 1.34e-3                    | 2.31e-3 | 1.22e-1          | 2.21e+0        | 130X130 |

Table 2: **The Respective First-order Approximation**

| $\varepsilon$ | $||u_\varepsilon - u_f^{0}||_2$ | UBE | $||\nabla u_\varepsilon - P^\varepsilon \nabla u_0||_2$ | $||K^\varepsilon \nabla u_\varepsilon - K^0 P^\varepsilon \nabla u_0||_2$ |
|----------------|----------------------------|------|--------------------------------|--------------------------------|
| (0.5)$^1$      | 1.34e-2                    | 3.65e-2 | 2.45e-1          | 3.47e+0        |
| (0.5)$^2$      | 8.56e-3                    | 2.17e-2 | 2.28e-1          | 3.35e+0        |
| (0.5)$^3$      | 4.57e-3                    | 1.21e-2 | 2.28e-1          | 3.26e+0        |
| (0.5)$^4$      | 2.89e-3                    | 5.92e-3 | 2.10e-1          | 2.96e+0        |
| (0.5)$^5$      | 1.09e-3                    | 2.31e-3 | 1.01e-1          | 2.57e+0        |
Generalized Geometries - Work in Progress...

\[
\begin{align*}
\text{Num}^* &= \begin{bmatrix} 1.91 & -0.101 \\ -0.101 & 1.91 \end{bmatrix} \\
\text{Anal} &= \begin{bmatrix} 1.86 & -0.102 \\ -0.102 & 1.86 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{Num}^i &= \begin{bmatrix} 5.33 & -0.286 \\ -0.286 & 6.761 \end{bmatrix} \\
\text{Anal}^i &= \begin{bmatrix} 5.73 & -0.212 \\ -0.212 & 6.55 \end{bmatrix}
\end{align*}
\]


*Amaziane et al. in: Comp. Geoscience (2001)*
Generalized Geometries - Work in Progress...
Generalized Geometries - Work in Progress...
Generalized Geometries - Work in Progress...

\[
\begin{align*}
\text{Num}^* &= \begin{bmatrix} 3.8292 & 0.1912 \\ 0.1912 & 2.7202 \end{bmatrix} \\
\text{Anal} &= \begin{bmatrix} 3.7713 & 0.0979 \\ 0.0979 & 2.7517 \end{bmatrix} \\
\text{Num}^* &= \begin{bmatrix} 0.2432 & -0.0101 \\ -0.0101 & 2.3542 \end{bmatrix} \\
\text{Anal} &= \begin{bmatrix} 0.2259 & -0.0137 \\ -0.0137 & 2.2065 \end{bmatrix} \\
\text{Num}^* &= \begin{bmatrix} 0.2060 & 0.0047 \\ 0.0047 & 0.1466 \end{bmatrix} \\
\text{Anal} &= \begin{bmatrix} 0.1922 & 0.0054 \\ 0.0054 & 0.1526 \end{bmatrix} \\
\text{Num}^* &= \begin{bmatrix} 3.4402 & 0.0000 \\ 0.0000 & 7.4172 \end{bmatrix} \\
\text{Anal} &= \begin{bmatrix} 3.5072 & 0.0029 \\ 0.0029 & 7.1862 \end{bmatrix} \\
\text{Num}^* &= \begin{bmatrix} 0.2179 & 0.0001 \\ 0.0001 & 4.3325 \end{bmatrix} \\
\text{Anal} &= \begin{bmatrix} 0.2640 & 0.0056 \\ 0.0056 & 4.1289 \end{bmatrix} \\
\text{Num}^* &= \begin{bmatrix} 1.6822 & 0.0479 \\ 0.0479 & 3.9697 \end{bmatrix} \\
\text{Anal} &= \begin{bmatrix} 1.7663 & 0.03608 \\ 0.0368 & 3.4894 \end{bmatrix}
\end{align*}
\]

\[\text{Num}^* - \text{Amaziane et al. in: Comp. Geoscience (2001)}.\]
Upscaling the Transport Equation

\[
\begin{align*}
\n\n\begin{cases}
\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n}\end{cases}
\end{align*}
\]

Where $D$ is the diffusivity $[L^2/T]$ and $f(x)$ is a pulse-type loading:

\[
\begin{align*}
f(x) = \begin{cases}
1 & t_0 < t_a \leq t_1 \\
0 & otherwise
\end{cases}
\end{align*}
\]

And $v^0(x)$ is the **Upscaled Darcy’s velocity**
Upscaling the Transport Equation
Pulse’s Response with Time

Concentration at the right corner from a unit pulse on the left corner

- Upscaled
- 256 inc
- 64 inc
- 16 inc
- 4 inc
Pulse’s Response with Time

Upscaled is accurate for computing total concentration
Pulse’s Response with Time

With twice the flow rate

\[ \frac{C}{C_0} \] vs. time (sec)

2.5 2.55 2.6 2.65 2.7 2.75 2.8 2.85
Conclusions - Future Work

- An analytical upscaling method was presented having the advantage of being computationally attractive and portable.
- The method applies to analogous Diffusion systems.
- The method applies to random media.
- Extension to Flow in Deformable Media is possible.
- Extension to Reaction-Diffusion equations is work in progress.
- Matching the results with experiments in an ongoing work.